Part 2： Simultaneous Equation Models (SEM)

Chapter 17. Endogeneity and Instrumental Variables

Chapter 18. Why Should We Concern SEM ?

Chapter 19. What is the Identification Problem ?

Chapter 20. How to Estimate SEM ?

Chapter 19. What is the Identification Problem ?

19.1 Identification Problem

Identification problem of SEM

The **identification problem** means whether numerical estimates of the **structural parameters** can be obtained from the estimated **reduced coefficients**.

Identification status:

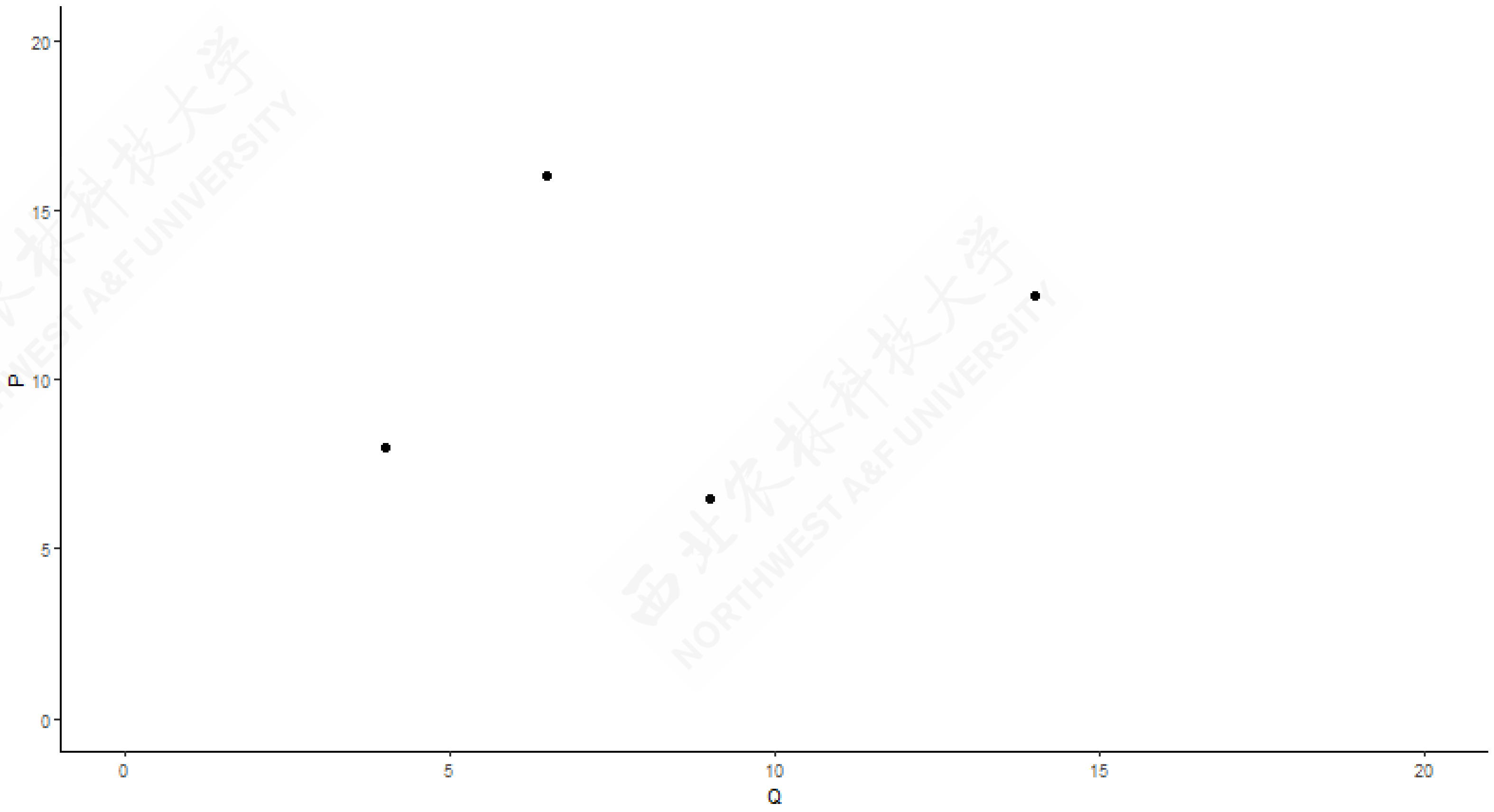
- If the estimator of structural parameter can be obtained, the equation is said to be **Identified**, and there may be two district situations:
  - **Just/Exact Identification**: The unique estimator of the structure parameter can be obtained.
  - **Overidentification**: More than one estimator of a structural parameter can be obtained.
- If the estimator of structural parameter cannot be obtained, the particular equation is said to be **Underidentification**.

Example: the structural SEM

Consider the following SEM for supply and demand:

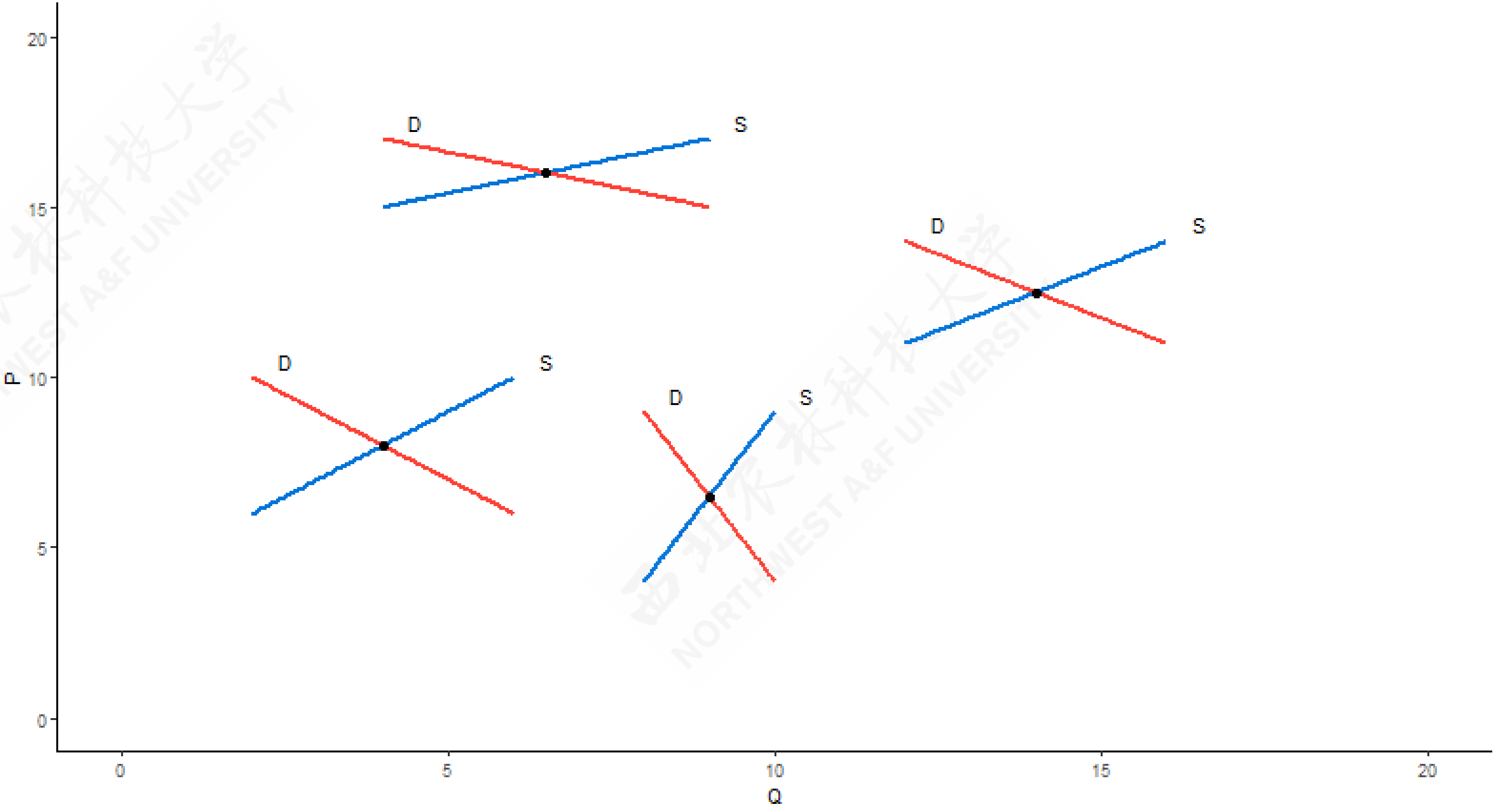
$$\begin{cases} Q = \alpha_0 + \alpha_1 P_t + u_{t1} & (\alpha_1 < 0) & \text{(demand function)} \\ Q = \beta_0 + \beta_1 P_t + u_{t2} & (\beta_1 > 0) & \text{(supply function)} \end{cases}$$

Example: scatter plots



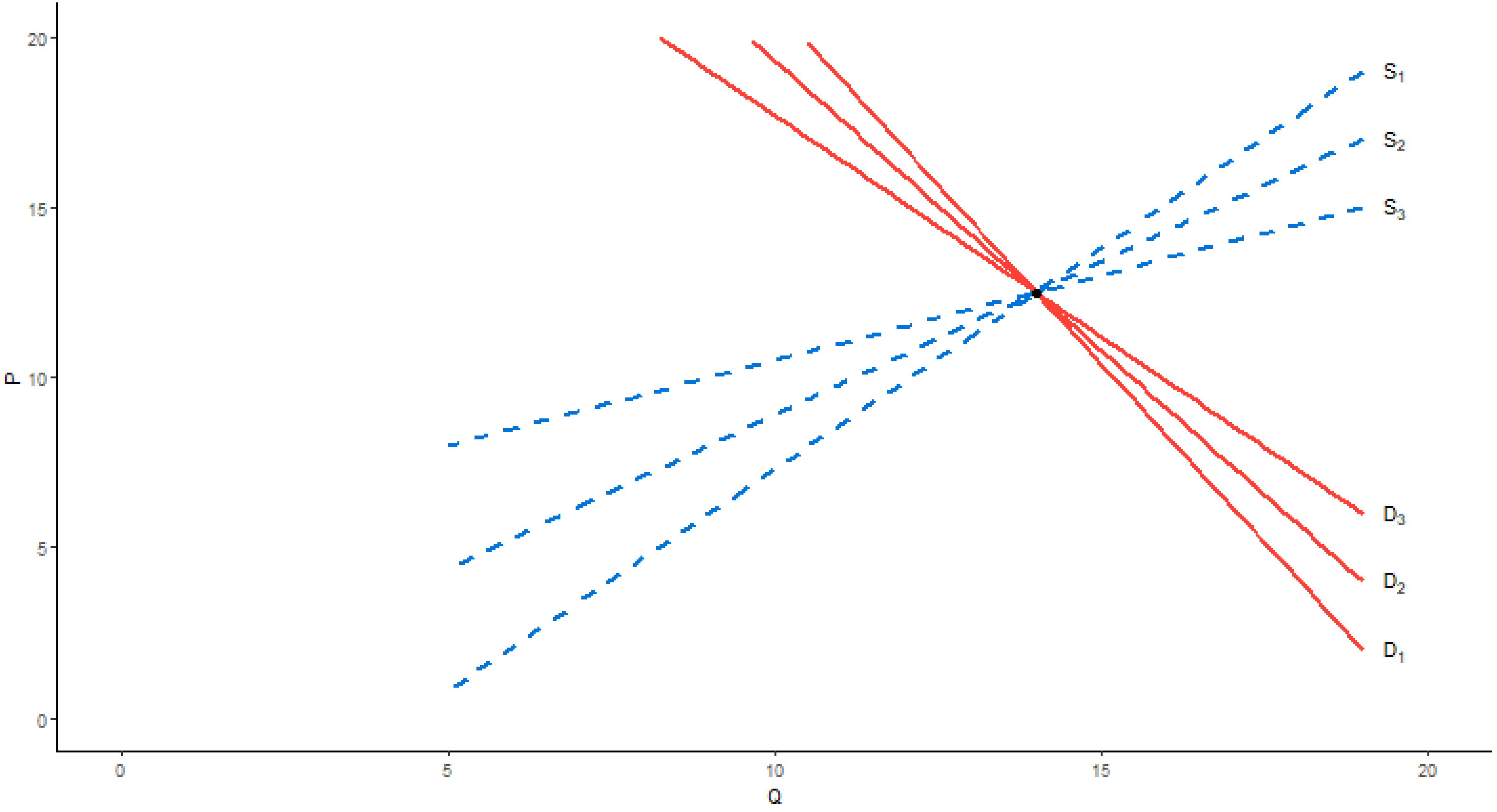
Scatter of Q and P

Example: equilibrium point of supply and demand



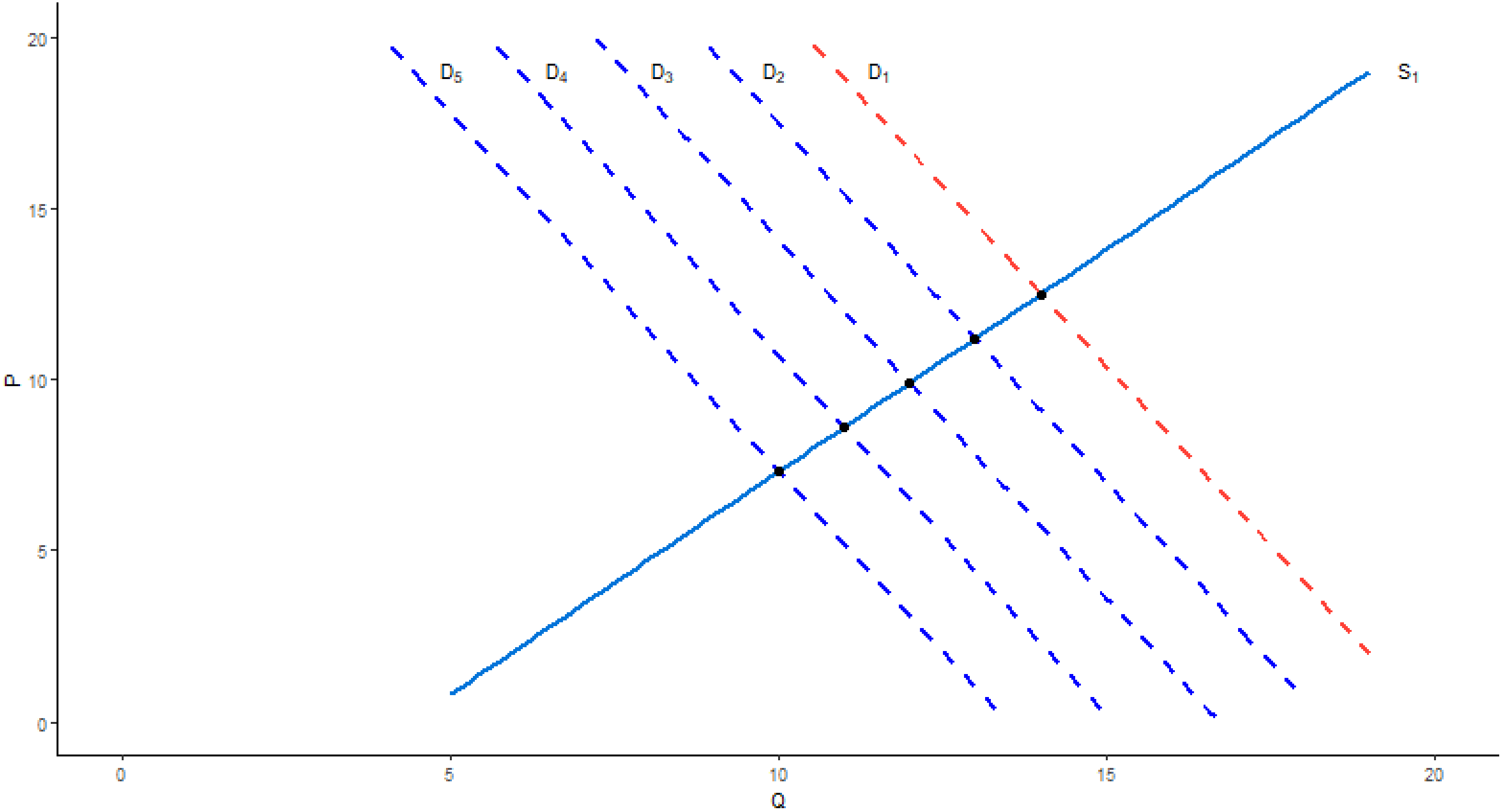
Each scatter represents the intersection of a demand curve and a supply curve.

Example: underidentification



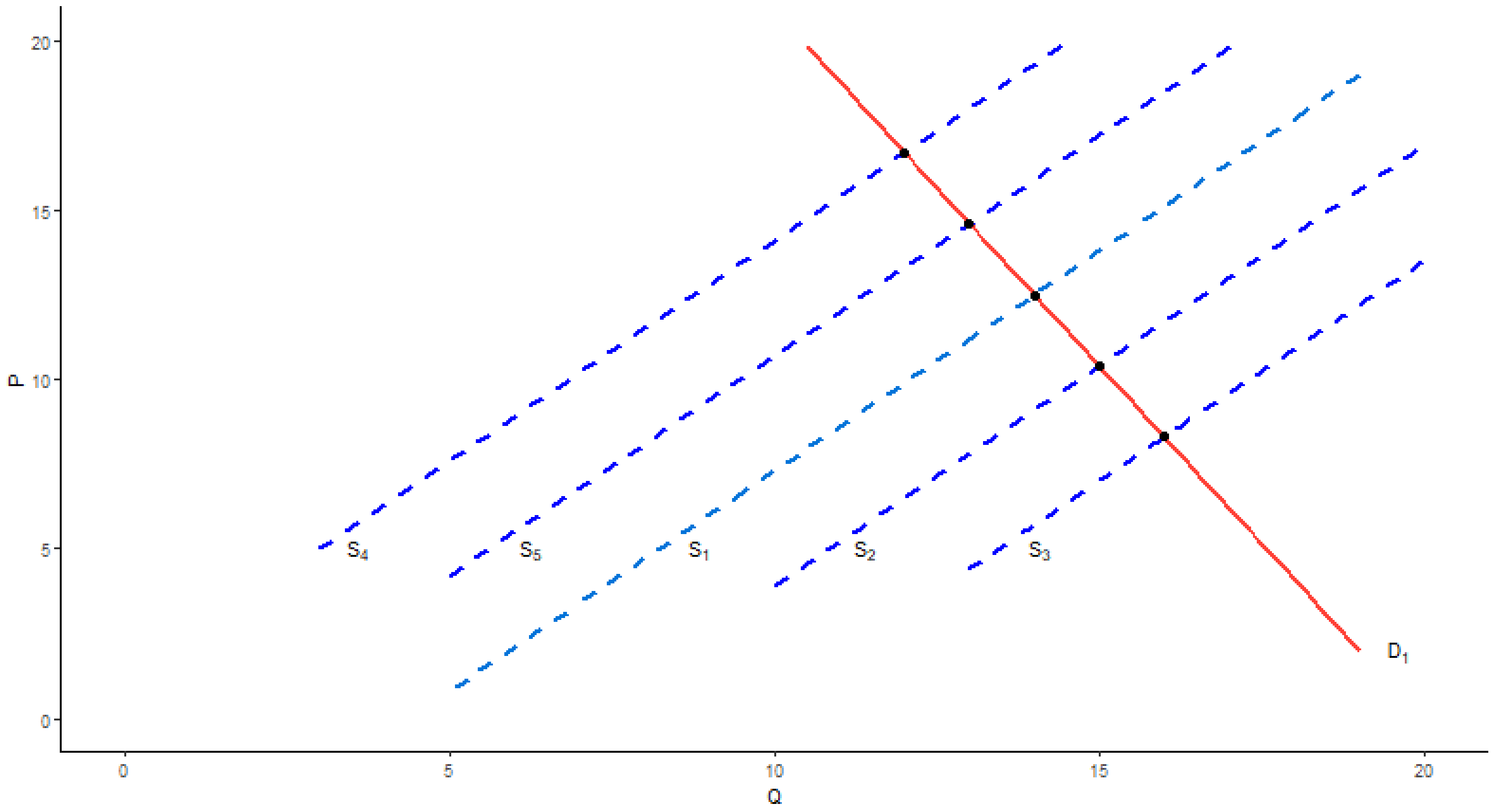
We can't be sure which of the entire family of supply and demand curves is responsible for this point. thus the supply and demand equation are both **underidentification**.

Example: the supply curve can be identified



If the demand curve shifts over time due to changes in income, tastes, etc., and the supply curve remains relatively stable, the scatter will show a **Supply Curve**. In this case, we say that the supply curve is **Exact Identification**.

Example: the demand curve can be identified



If the supply curve shifs over time due to changes in climatic conditions or other external factors, but the demand curve remains relatively stable, the scatter will show a **demand curve**. In this case, we say that the demand curve is **identified**.

Under identification: Structural and reduced SEM

Give the Structural SEM:

$$\begin{cases} Q_t = \alpha_0 + \alpha_1 P_t + u_{t1} & (\alpha_1 < 0) \quad \text{(demand function)} \\ Q_t = \beta_0 + \beta_1 P_t + u_{t2} & (\beta_1 > 0) \quad \text{(supply function)} \end{cases}$$

> It's easy to obtain the Reduced SEM:

$$\begin{cases} P_t = \pi_{11} + v_{t1} \\ Q_t = \pi_{12} + v_{t2} \end{cases}$$

> Obviously, the structural SEM is **underidentification**! (Why?)
>
> - number of structural parameters? 2
> - number of reduced parameters? 0

Just identification: Structural and reduced SEM

Given **Structural** SEM (I = income of the consumer):

$$\begin{cases} Q_t = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + u_{t1} & (\alpha_1 < 0, \alpha_2 > 0) \quad \text{(demand function)} \\ Q_t = \beta_0 + \beta_1 P_t + u_{t2} & (\beta_1 > 0) \quad \text{(supply function)} \end{cases}$$

We can obtain the **Reduced** SEM:

$$\begin{cases} P_t = \pi_{11} + \pi_{21} I_t + v_{t1} \\ Q_t = \pi_{12} + \pi_{22} I_t + v_{t2} \end{cases}$$

$$\begin{cases} \pi_{11} = \dfrac{\beta_0 - \alpha_0}{\alpha_1 - \beta_1}; & \pi_{21} = -\dfrac{\alpha_2}{\alpha_1 - \beta_1}; & v_{t1} = \dfrac{u_{t2} - u_{t1}}{\alpha_1 - \beta_1} \\ \pi_{12} = \dfrac{\alpha_1 \beta_0 - \alpha_0 \beta_1}{\alpha_1 - \beta_1}; & \pi_{22} = -\dfrac{\alpha_2 \beta_1}{\alpha_1 - \beta_1}; & v_{t2} = \dfrac{\alpha_1 u_{t2} - \beta_1 u_{t1}}{\alpha_1 - \beta_1} \end{cases}$$

Just identification: part equation just-identified

Question: Can the aforementioned structural equation be identified?

- number of reduced parameters?
- number of structural parameters?

Answer:

- Only 4 reduced parameters: $\pi_{11}, \pi_{21}, \pi_{12}, \pi_{22}$
- But 5 structural parameters: $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1$

Therefore, it is impossible to completely solve all 5 structural parameters.

However, the **supply equation** is **Exact identification**, because:

$$\beta_0 = \pi_{12} - \beta_1 \pi_{11}; \quad \beta_1 = \frac{\pi_{22}}{\pi_{21}}$$

But there is no unique way of estimating the parameters of the **demand function**.

Therefore, the **demand equation** remains underidentified.

Just identification: Structural SEM

Given **Structural** SEM (P_{t-1} = price lagged one period):

$$\begin{cases} Q_t = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + u_{t1} & (\alpha_1 < 0, \alpha_2 > 0) \quad \text{(demand function)} \\ Q_t = \beta_0 + \beta_1 P_t + \beta_2 P_{t-1} + u_{t2} & (\beta_1 > 0, \beta_2 > 0) \quad \text{(supply function)} \end{cases}$$

> where the demand function remains as before but the supply function includes an additional explanatory variable, price lagged one period.

We can obtain the **Reduced** SEM:

$$\begin{cases} P_t = \pi_{11} + \pi_{21} I_t + \pi_{31} P_{t-1} + v_{t1} \\ Q_t = \pi_{12} + \pi_{22} I_t + \pi_{32} P_{t-1} + v_{t2} \end{cases}$$

And obtain the relationship between the structural coefficients and the reduced coefficients.

$$\begin{cases} \pi_{11} = \frac{\beta_0 - \alpha_0}{\alpha_1 - \beta_1}; \quad \pi_{21} = -\frac{\alpha_2}{\alpha_1 - \beta_1}; \quad \pi_{31} = -\frac{\beta_2}{\alpha_1 - \beta_1}; \quad v_{t1} = \frac{u_{t2} - u_{t1}}{\alpha_1 - \beta_1} \\ \pi_{12} = \frac{\alpha_1 \beta_0 - \alpha_0 \beta_1}{\alpha_1 - \beta_1}; \quad \pi_{22} = -\frac{\alpha_2 \beta_1}{\alpha_1 - \beta_1}; \quad \pi_{32} = \frac{\alpha_1 \beta_2}{\alpha_1 - \beta_1}; \quad v_{t2} = \frac{\alpha_1 u_{t2} - \beta_1 u_{t1}}{\alpha_1 - \beta_1} \end{cases}$$

> So, can we calculate the unique structural coefficients by using the numerical reduced coefficients?

Just identification: all equations just-identified

Question： Can the aforementioned structural equation be identified?

- number of reduced parameters?
- number of structural parameters?

Answer：

- there are 6 reduced parameters: $\pi_{11}, \pi_{21}, \pi_{31}; \pi_{12}, \pi_{22}, \pi_{32}$
- and 6 structural parameters: $\alpha_0, \alpha_1, \alpha_2; \beta_0, \beta_1, \beta_2$

> Therefore, the parameters of both the SEM equations can be identified, and the system as a whole can be identified exactly.

Over-identification: Structural SEM

Now suppose we consider the following structural SEM for demand-and-supply:

$$\begin{cases} Q_t = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + \alpha_3 R_t + u_{t1} & (\alpha_1 < 0, \alpha_2 > 0) \quad \text{(demand function)} \\ Q_t = \beta_0 + \beta_1 P_t + \beta_2 P_{t-1} + u_{t2} & (\beta_1, \beta_2 > 0) \quad \text{(supply function)} \end{cases}$$

> Where the supply function remains as before but the demand function includes two additional explanatory variables, **income**(I_t) and **wealth**(R_t).

> **Questions**: Can you transform the structural SEM to get the reduced SEM?

Over-identification: Reduced SEM

We can get the reduced SEM and the relationship between Structural and reduced pars:

$$\begin{cases} P_t = \pi_{11} + \pi_{21} I_t + \pi_{31} R_t + \pi_{41} P_{t-1} + v_{t1} \\ Q_t = \pi_{12} + \pi_{22} I_t + \pi_{32} R_t + \pi_{42} P_{t-1} + v_{t2} \end{cases}$$

$$\begin{cases} \pi_{11} = \dfrac{\beta_0 - \alpha_0}{\alpha_1 - \beta_1} \\ \pi_{21} = -\dfrac{\alpha_2}{\alpha_1 - \beta_1} \\ \pi_{31} = -\dfrac{\alpha_3}{\alpha_1 - \beta_1} \\ \pi_{41} = \dfrac{\beta_2}{\alpha_1 - \beta_1} \\ v_{t1} = \dfrac{u_{t2} - u_{t1}}{\alpha_1 - \beta_1} \end{cases}
\qquad
\begin{cases} \pi_{12} = -\dfrac{\alpha_1\beta_0 - \alpha_0\beta_1}{\alpha_1 - \beta_1} \\ \pi_{22} = -\dfrac{\alpha_2\beta_1}{\alpha_1 - \beta_1} \\ \pi_{32} = -\dfrac{\alpha_3\beta_1}{\alpha_1 - \beta_1} \\ \pi_{42} = \dfrac{\alpha_1\beta_2}{\alpha_1 - \beta_1} \\ v_{t2} = \dfrac{\alpha_1 u_{t2} - \beta_1 u_{t1}}{\alpha_1 - \beta_1} \end{cases}$$

Over-identification: multiple solutions

Questions:

Can the Structural SEM be identified?

- number of reduced parameters?
- number of structural parameters?

Answer:

- 8 **Reduced parameters** :

  $\pi_{11}, \pi_{12}, \pi_{13}, \pi_{14}, \pi_{21}, \pi_{22}, \pi_{23}, \pi_{24}$
- Only 7 **structural Parameters** :

  $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2$

Therefore, the number of equations is more than the number of unknown coefficients, so the unique estimation value of all 7 structural coefficients cannot be obtained.

> **Conclusion**: there are many solutions to the structural SEM that satisfy the condition. So it is **Overidentification**.

19.2 Identification rules

Symbols and notations

Firstly, let us define the notation for the number of **variables** in the **Structural SEM**.

- M = The number of all **endogenous variables** in the structural SEM.
- K = The number of all **predetermined variables** in the structural SEM (including the intercept term)
- m = The number of endogenous variables in a particular equation.
- k = The number of **predeterminate variables** in a particular equation (including the intercept term).

The order rules: solution 1

A necessary (but not sufficient) condition of identification, known as the **order condition**, may be stated in two different but equivalent ways.

Order rules 1 :

> In a model of M simultaneous equations, in order for an equation to be identified, it must exclude at least $M-1$ variables (endogenous as well as predetermined) appearing in the SEM.
>
> - If it excludes exactly $M-1$ variables, the equation is **just identified**. Which means $M+K-(k+m)=(M-1)$.
> - If it excludes more than $M-1$ variables, it is **overidentified**. Which means $M+K-(k+m)>(M-1)$

The order rules: solution 2

And here is another equivalent order rule.

Order rules 2:

> In a model of M simultaneous equations, in order for an equation to be identified, the number of predetermined variables(k) excluded from the equation must not be less than the number of endogenous variables(m) included in that equation less 1, that is,
>
> - If $K - k = m - 1$, the equation is **just identified**.
> - but if $K - k > m - 1$, it is **overidentified**.

Case demo: both under-identification

The **structural SEM** is:

$$\begin{cases} Q_t = \alpha_0 + \alpha_1 P_t + u_{t1} & (\alpha_1 < 0) & \text{(demand function)} \\ Q_t = \beta_0 + \beta_1 P_t + u_{t2} & (\beta_1 > 0) & \text{(supply function)} \end{cases}$$

Conclusion from **Order Rule 1** :

- All numbers of variables in the structural SEM is $(M+K) = 2+1 = 3$, and $(M-1) = 2-1 = 2$.
- For the first equation, because number of all variables is $(m+k) = 2+1 = 3$, so $(M+K)-(m+k) = 3-3 = 0$. We see $(M+K)-(m+k) < (M-1) = 1$. So the first equation (the demand equation) is **Underidentification**.
- For the second equation, because $(m+k) = 2+1 = 3$, so $(M+K)-(m+k) = 3-3 = 0$. We see $(M+K)-(m+k) < (M-1) = 1$. So the second equation (the supply equation) is also **Underidentification**

Case demo: both under-identification

The **structural SEM** is:

$$\begin{cases} Q_t = \alpha_0 + \alpha_1 P_t + u_{t1} & (\alpha_1 < 0) & \text{(demand function)} \\ Q_t = \beta_0 + \beta_1 P_t + u_{t2} & (\beta_1 > 0) & \text{(supply function)} \end{cases}$$

Conclusion from **Order Rule 2** :

- The number of predetermined variables in the structural SEM is $K = 1$.
- For the **first** equation: the number of predetermined variables is $k = 1$, and $(K - k) = 0$; The number of endogenous variables is $m = 2$, and $(m - 1) = 1$; We see $(K - k) < (m - 1)$. So the first equation (the demand equation) is **Underidentification**.
- For the **second** equation: the number of predetermined variables is $k = 0$, and $(K - k) = 0$; The number of endogenous variables is $m = 1$, and $(m - 1) = 1$; We see $(K - k) < (m - 1)$. So the second equation (the supply equation) is also **Underidentification**.

Case demo: (Under + Just) identification

The **structural SEM** is:

$$\begin{cases} Q_t = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + u_{t1} & (\alpha_1 < 0, \alpha_2 > 0) \quad \text{(demand function)} \\ Q_t = \beta_0 + \beta_1 P_t + u_{t2} & (\beta_1 > 0) \quad \text{(supply function)} \end{cases}$$

Conclusion from **Order Rule 1** :

- In the structural SEM: The number of all variables is $(M + K) = 2 + 2 = 4$, and $(M - 1) = 2 - 1 = 1$.
- The first eq: the number of all variables is $(m + k) = 2 + 2 = 4$, so $(M + K) - (m + k) = 4 - 4 = 0$. We see $(M + K) - (m + k) < (M - 1) = 1$. Thus the first equation is **Underidentification**.
- The second eq: the number of all variables is $(m + k) = 2 + 1 = 3$, so $(M + K) - (m + k) = 4 - 3 = 1$. We see $(M + K) - (m + k) = (M - 1) = 1$. Thus the second equation is **Just identification**.

Case demo: (Under + Just) identification

The **structural SEM** is:

$$\begin{cases} Q_t = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + u_{t1} & (\alpha_1 < 0, \alpha_2 > 0) \quad \text{(demand function)} \\ Q_t = \beta_0 + \beta_1 P_t + u_{t2} & (\beta_1 > 0) \quad \text{(supply function)} \end{cases}$$

Conclusion from **Order Rule 2** :

- The number of predetermined variables in the structural SEM is $K = 2$
- For the first equation: The number of predetermined variables is $k = 2$, and $(K - k) = 0$; The number of endogenous variables is $m = 2$, and (m-1)=1; Obviously $(K - k) < (m - 1) = 1$. So the first equation (the demand equation) is **Underidentification**.
- For the second equation: the number of predetermined variables is $k = 1$, and $(K - k) = 1$; The number of endogenous variables is $m = 2$, and $(m - 1) = 1$; Obviously $(K - k) = (m - 1) = 1$. So the second equation (supply equation) is **Just identification**.

Case demo: (Just + Over) identification

The **structural SEM** is:

$$\begin{cases} Q_t = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + \alpha_3 R_t + u_{t1} & (\alpha_1 < 0, \alpha_2 > 0) \quad \text{(demand function)} \\ Q_t = \beta_0 + \beta_1 P_t + \beta_2 P_{t-1} + u_{t2} & (\beta_1, \beta_2 > 0) \quad \text{(supply function)} \end{cases}$$

Conclusion from **Order Rule 1** :

- In this structural SEM: the number of all variables is $(M + K) = 2 + 4 = 6$, and $(M - 1) = 2 - 1 = 1$
- The first equation: the number of all variables is $(m + k) = 2 + 3 = 5$, so $(M + K) - (m + k) = 6 - 5 = 1$. We see clearly $(M + K) - (m + k) = (M - 1) = 1$. Hence the first equation is **Just Identification**
- The second equation: the number of all variables is $(m + k) = 2 + 2 = 4$, so that $(M + K) - (m + k) = 6 - 4 = 2$. We see $(M + K) - (m + k) > (M - 1) = 1$. Thus, the second equation is

Case demo: (Just + Over) identification

The **structural SEM** is:

$$\begin{cases} Q_t = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + \alpha_3 R_t + u_{t1} & (\alpha_1 < 0, \alpha_2 > 0) & \text{(demand function)} \\ Q_t = \beta_0 + \beta_1 P_t + \beta_2 P_{t-1} + u_{t2} & (\beta_1, \beta_2 > 0) & \text{(supply function)} \end{cases}$$

Conclusion from **Order Rule 2** :

- In the structural SEM: The number of predetermined variables $K = 4$.
- The first equation: the number of predetermined variables $k = 3$, and $(K - k) = 1$; The number of endogenous variables is $m = 2$, and $(m - 1) = 1$; We see $(K - k) = (m - 1) = 1$. So the first equation is the **Just Identification**.
- The second equation: the number of predetermined variables is $k = 2$, and $(K - k) = 2$; The number of endogenous variables is $m = 2$, and $(m - 1) = 1$; We see $(K - K) > (m - 1) = 1$. So the second equation is **overidentification**.

The order rules: Conclusion

- Firstly, the order rule is a **necessary** but not **sufficient** condition for Identificatoin problems. Thus even if order conditions are satisfied, the equation will be unidentifiable.

> See the following example:

$$\begin{cases} Q = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + u_{t1} & (\alpha_1 < 0, \alpha_2 > 0) \quad \text{(demand function)} \\ Q = \beta_0 + \beta_1 P_t + u_{t2} & (\beta_1 > 0) \quad \text{(supply function)} \end{cases}$$

> According to the rules of order conditions, we can judge that the second equation (the supply equation) is **Just Identification**.
> But in fact, we also need to make sure that the coefficient of income variable I_t in the first equation should satisfy $\alpha_2 \neq 0$.
> But notice that the zero restrictions criterion is based on a priori or theoretical expectations that certain variables do not appear in a given equation.

The order rules： Conclusion

- Secondly, **Order Rule 1** and **Order Rule 2** are equivalent.
- Finally, it's only suitable for the simple SEM situatons to use the Rule of Order Condition.

The Rank Rules: main procedure

The Rank Rules:

> In a model containing M equations in M endogenous variables, an equation is identified if and only if at least one non-zero determinant of order $(M-1) \times (M-1)$ can be constructed from the coefficients of the variables (both endogenous and predetermined) excluded from that particular equation but included in the other equations of the model.

Here are the **Main proceeding steps**:

> 1. Transform the structural SEM and write down the **algebraic formula 2**.
> 2. Write the corresponding table of **equation coefficients**.
> 3. Find all **columns** that are not included in the equation.
> 4. Construct any $(M-1) * (M-1)$ matrix among these columns.
> 5. Judge and draw a conclusion: if any matrix with a determinant of 0 can be found, the equation is **underidentification**.

Step 1: fill table form of coefficients

$$
\begin{cases}
Y_{t1} - \gamma_{21}Y_{t2} - \gamma_{31}Y_{t3} - \beta_{01} - \beta_{11}X_{t1} = u_{t1} \\
Y_{t2} - \gamma_{32}Y_{3t} - \beta_{02} - \beta_{12}X_{1t} - \beta_{22}X_{2t} = u_{t2} \\
-\gamma_{13}Y_{t1} + Y_{t3} - \beta_{03} - \beta_{13}X_{1t} - \beta_{23}X_{2t} = u_{t3} \\
-\gamma_{14}Y_{t1} - \gamma_{24}Y_{t2} + Y_{t4} - \beta_{04} - \beta_{34}X_{t3} = u_{t4}
\end{cases}
$$

The parameters of the structural SEM can be written as follows:

eq	Y_1	Y_2	Y_3	Y_4	1	X_1	X_2	X_3
1	1	$-\gamma_{21}$	$-\gamma_{31}$	0	$-\beta_{01}$	$-\beta_{11}$	0	0
2	0	1	$-\gamma_{32}$	0	$-\beta_{02}$	$-\beta_{12}$	$-\beta_{22}$	0
3	$-\gamma_{13}$	0	1	0	$-\beta_{03}$	$-\beta_{13}$	$-\beta_{23}$	0
4	$-\gamma_{14}$	$-\gamma_{24}$	0	1	$-\beta_{04}$	0	0	$-\beta_{34}$

Step 2: check variables and column

$$\begin{cases} Y_{t1} - \gamma_{21}Y_{t2} - \gamma_{31}Y_{t3} - \beta_{01} - \beta_{11}X_{t1} = u_{t1} \\ Y_{t2} - \gamma_{32}Y_{3t} - \beta_{02} - \beta_{12}X_{1t} - \beta_{22}X_{2t} = u_{t2} \\ -\gamma_{13}Y_{t1} + Y_{t3} - \beta_{03} - \beta_{13}X_{1t} - \beta_{23}X_{2t} = u_{t3} \\ -\gamma_{14}Y_{t1} - \gamma_{24}Y_{t2} + Y_{t4} - \beta_{04} - \beta_{34}X_{t3} = u_{t4} \end{cases}$$

- The endogenous variables not included in the first equation are: Y_{t4};and predetermined variables not included: X_{t2}, X_{t3}.

eq	Y_1	Y_2	Y_3	$[Y_4]$	1	X_1	$[X_2]$	$[X_3]$
1	1	$-\gamma_{21}$	$-\gamma_{31}$	0	$-\beta_{01}$	$-\beta_{11}$	0	0
2	0	1	$-\gamma_{32}$	0	$-\beta_{02}$	$-\beta_{12}$	$-\beta_{22}$	0
3	$-\gamma_{13}$	0	1	0	$-\beta_{03}$	$-\beta_{13}$	$-\beta_{23}$	0
4	$-\gamma_{14}$	$-\gamma_{24}$	0	1	$-\beta_{04}$	0	0	$-\beta_{34}$

Step 3: obtain matrix determinant

eq	Y_1	Y_2	Y_3	$[Y_4]$	1	X_1	$[X_2]$	$[X_3]$
1	1	$-\gamma_{21}$	$-\gamma_{31}$	0	$-\beta_{01}$	$-\beta_{11}$	0	0
2	0	1	$-\gamma_{32}$	0	$-\beta_{02}$	$-\beta_{12}$	$-\beta_{22}$	0
3	$-\gamma_{13}$	0	1	0	$-\beta_{03}$	$-\beta_{13}$	$-\beta_{23}$	0
4	$-\gamma_{14}$	$-\gamma_{24}$	0	1	$-\beta_{04}$	0	0	$-\beta_{34}$

$$A = \begin{bmatrix} 0 & -\beta_{22} & 0 \\ 0 & -\beta_{32} & 0 \\ 1 & 0 & -\beta_{34} \end{bmatrix} \qquad det(A) = \begin{vmatrix} 0 & -\beta_{22} & 0 \\ 0 & -\beta_{32} & 0 \\ 1 & 0 & -\beta_{43} \end{vmatrix} = 0$$

That means the rank of the matrix $rank(A) = \rho(A) < 3$. Therefore, the first equationdoes does not satisfy the **rank condition**, and it is **underidentification**.

The Rank Rules: Conclusion

In summary, the steps of the rank condition rules are as follows:

- Write down the **structural SEM** in the **algebra form 2**;
- Put the **coefficients** in tabular form;
- Strike out the coefficients of the row in which the equation under consideration;
- Strike out the columns of the non-zero coefficients in the equation considered
- The remaining coefficients will form a **coefficient matrix**
- Construct arbitrary matrix with $(M-1)*(M-1)$ and calculate the determinant.

> - If there is at least one square matrix $(M-1)*(M-1)$, with which determinant is not equal to zero (namely, the rank of the square matrix is $m-1$), it will indicates that the equation under consideration is **Just Identification**.
> - If determinants of all possible square matrices $(M-1)*(M-1)$are all equal to zero, which means the rank of all these square matrix is less than $M-1$, the equation considered is **Underidentification**.

Compare: results from the rank rules

eq	Y_1	Y_2	Y_3	Y_4	1	X_1	X_2	X_3
1	1	$-\gamma_{21}$	$-\gamma_{31}$	0	$-\beta_{01}$	$-\beta_{11}$	0	0
2	0	1	$-\gamma_{32}$	0	$-\beta_{02}$	$-\beta_{12}$	$-\beta_{22}$	0
3	$-\gamma_{13}$	0	1	0	$-\beta_{03}$	$-\beta_{13}$	$-\beta_{23}$	0
4	$-\gamma_{14}$	$-\gamma_{24}$	0	1	$-\beta_{04}$	0	0	$-\beta_{34}$

Answer:

- Equation 1: **underidentification**
- Equation 2: ? **underidentification**
- Equation 3: ? **underidentification**
- Equation 4: ? **identified**

Compare: results from the order rules

$$
\begin{cases}
Y_{t1} - \gamma_{21}Y_{t2} - \gamma_{31}Y_{t3} - \beta_{01} - \beta_{11}X_{t1} = u_{t1} \\
Y_{t2} - \gamma_{32}Y_{3t} - \beta_{02} - \beta_{12}X_{1t} - \beta_{22}X_{2t} = u_{t2} \\
-\gamma_{13}Y_{t1} + Y_{t3} - \beta_{03} - \beta_{13}X_{1t} - \beta_{23}X_{2t} = u_{t3} \\
-\gamma_{14}Y_{t1} - \gamma_{24}Y_{t2} + Y_{t4} - \beta_{04} - \beta_{34}X_{t3} = u_{t4}
\end{cases}
$$

With order rules 2, you should obtain following conclusion for all 4 equation:

eq	m	k	M	K	K-k	m-1	result
1	3	2	4	4	2	2	just identification
2	2	3	4	4	1	1	just identification
3	2	3	4	4	1	1	just identification
4	3	2	4	4	2	2	just identification

Summary on identification rules

The following is a comprehensive summary of the identification rules:

- The order condition is a necessary but not sufficient condition for the identification ploblem, and even if it is satisfied, the equation still may be unidentifiable.
- The rank condition is the **sufficient and necessary conditions** for identification problem.
- Rank rules can tell us that the if the equations is identifible or unidentifiable. While the order rules conditions will tell us if it is **just identification** or **over identification**.
- Strictly, we need to first analyze the rank rules to determine whether the equation is identifiable; And then use the order rules to judge if it is **just identification** or **over identification**.

19.3 Endogeneity test (Test of Simultaneity)*

Because we have learn these techniques in chapter 17, so we will jump to the chapter 20.

Endogeneity test: Concepts and definitions

Simultaneity testing is essentially testing whether a (endogenous) regressor is related to the error term (u_t).

- if relevant, there is a problem of simultaneity, so we need to find other estimation method different from OLS;
- if not, It will seems that there is no problem of simultaneity and you can use the OLS method as usual.

Test of Simultaneity also known as **Hausman test of endogeneity**.

- **Hausman specification error test** can be used to test simultaneous problems.

Endogeneity test: the theory principle

Given the structural SEM:

$$Q = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + \alpha_3 R_t + u_{t1} \qquad (\alpha_1 < 0, \alpha_2 > 0) \qquad \text{(demand function)}$$
$$Q = \beta_0 + \beta_1 P_t + u_{t2} \qquad (\beta_1 > 0) \qquad \text{(supply function)}$$

where: P=price; I=income; R=wealth

It is easy to get the reduced SEM:

$$P_t = \pi_{11} + \pi_{21} I_t + \pi_{31} R_t + v_{t1} \qquad \text{(eq1)}$$
$$Q_t = \pi_{12} + \pi_{22} I_t + \pi_{33} R_t + v_{t2} \qquad \text{(eq2)}$$

Endogeneity test: the theory principle

We can directly estimate reduced equation 1 (price equation) by OLS method:

$$
\begin{aligned}
P_t &= \hat{\pi}_{11} + \hat{\pi}_{21} I_t + \hat{\pi}_{31} R_t + \hat{v}_{t1} \qquad \text{(OLS Estimation)} \\
&= \hat{P}_t + \hat{v}_{t1}
\end{aligned}
$$

Then, we can use the OLS estimation results to construct two **test equations**:

$$
\begin{aligned}
Q_t &= \beta_0 + \beta_1 \hat{P}_t + \beta_1 \hat{v}_{t1} + u_{t2} \qquad \text{(Hausman test equation)} \\
Q_t &= \beta_0 + \beta_1 P_t + \beta_1 \hat{v}_{t1} + u_{t2} \qquad \text{(Pindyck test equation)}
\end{aligned}
$$

> Hausman test:
>
> - Null hypothesis H_0 : No simultaneity problems, so that $\hat{v}_{t1}$ is uncorrelated with u_{t2};
> - Alternative hypothesis H_1 : exist simultaneity problems, so that $\hat{v}_{t1}$ is correlated with u_{t2};

We can directly estimate reduced equation 1 (price equation) by OLS method:

$$\begin{aligned} P_t &= \hat{\pi}_{11} + \hat{\pi}_{21} I_t + \hat{\pi}_{31} R_t + \hat{v}_{t1} \qquad \text{(OLS Estimation)} \\ &= \hat{P}_t + \hat{v}_{t1} \end{aligned}$$

Then, we can use the OLS estimation results to construct two **test equations**:

$$\begin{aligned} Q_t &= \beta_0 + \beta_1 \hat{P}_t + \beta_1 \hat{v}_{t1} + u_{t2} \qquad \text{(Hausman test equation)} \\ Q_t &= \beta_0 + \beta_1 P_t + \beta_1 \hat{v}_{t1} + u_{t2} \qquad \text{(Pindyck test equation)} \end{aligned}$$

Thus, we only need to test β_1 in the Hausman test.

- if the test is **not significant**, H_0 can not be rejected, and conclude No exist of simultaneity problems.
- if the test is **significant**, H_0 should be rejected, and conclude the exist of simultaneity problems.

Endogeneity test: Steps

The steps of Hausman test include:

- step 1: conduct the fist OSL estimation $P_t = \hat{\pi}_{11} + \hat{\pi}_{21} I_t + \hat{\pi}_{31} R_t + \hat{v}_{t1}$, and obtain its residuals $\hat{v}_t$
- step 2： conduct the second OSL estimation $Q_t = \beta_0 + \beta_1 \hat{P}_t + \beta_1 \hat{v}_{t1} + u_{t2}$, and apply t test for $\hat{v}_{1t}$,then judge according former rules.

> Note: To esitimat more efficiently, Pindyck and Rubinfeld suggest the second OSL estimation shoild be: $Q_t = \beta_0 + \beta_1 P_t + \beta_1 \hat{v}_{t1} + u_{t2}$

Example: Structural and reduced SEM

In the truffle case, given Structural SEM：

$$\begin{cases} Q_i = \alpha_0 + \alpha_1 P_i + \alpha_2 PS_i + \alpha_3 DI_i + u_{i1} & \text{(demand function)} \\ Q_i = \beta_0 + \beta_1 P_i + \beta_2 PF_i + u_{i2} & \text{(supply function)} \end{cases}$$

And obtain its reduced SEM：

$$\begin{cases} P_i = \pi_{11} + \pi_{21} PS_i + \pi_{31} DI_i + \pi_{41} PF_i + v_{i1} \\ Q_i = \pi_{12} + \pi_{22} PS_t + \pi_{32} DI_t + \pi_{42} PF_i + v_{i2} \end{cases}$$

Example: Structural and reduced coefficients

The relationship between **reduced coefficients** and **structural coefficients** is:

$$\begin{cases} \pi_{11} = \dfrac{\beta_0 - \alpha_0}{\alpha_1 - \beta_1} \\ \pi_{21} = -\dfrac{\alpha_2}{\alpha_1 - \beta_1} \\ \pi_{31} = -\dfrac{\alpha_3}{\alpha_1 - \beta_1} \\ \pi_{41} = \dfrac{\beta_2}{\alpha_1 - \beta_1} \\ v_{i1} = \dfrac{u_{i2} - u_{i1}}{\alpha_1 - \beta_1} \end{cases}$$

$$\begin{cases} \pi_{12} = -\dfrac{\alpha_1\beta_0 - \alpha_0\beta_1}{\alpha_1 - \beta_1} \\ \pi_{22} = -\dfrac{\alpha_2\beta_1}{\alpha_1 - \beta_1} \\ \pi_{32} = -\dfrac{\alpha_3\beta_1}{\alpha_1 - \beta_1} \\ \pi_{42} = \dfrac{\alpha_1\beta_2}{\alpha_1 - \beta_1} \\ v_{i2} = \dfrac{\alpha_1 u_{i2} - \beta_1 u_{i1}}{\alpha_1 - \beta_1} \end{cases}$$

Example: First OLS estimation on price equation

According to the first step, OLS method is used to estimate the **reduced price equation** and obtain $\hat{P}_i, \hat{v}_{i1}$

> The **reduced price equation** :

$$
\begin{aligned}
P_i &= \hat{\pi}_{11} + \hat{\pi}_{21} PS_i + \hat{\pi}_{31} DI_i + \hat{\pi}_{41} PF_i + \hat{v}_{i1} \\
&= \hat{P}_i + \hat{v}_{i1}
\end{aligned}
$$

Example: First OLS estimation on price equation

The raw R report for the OLS estimation of **reduced price equation** :

```
Call:
lm(formula = Hausman_models$mod.P, data = truffles)

Residuals:
   Min     1Q Median     3Q    Max 
-20.48  -3.59   0.28   4.53  12.92 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  -32.512      7.984   -4.07  0.00039 ***
PS             1.708      0.351    4.87  4.8e-05 ***
DI             7.602      1.724    4.41  0.00016 ***
PF             1.354      0.299    4.54  0.00011 ***
---
Signif. codes:  
0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 6.6 on 26 degrees of freedom
Multiple R-squared:  0.889,	Adjusted R-squared:  0.876 
F-statistic: 69.2 on 3 and 26 DF,  p-value: 1.6e-12
```

Example: First OLS estimation on price equation

The tidy R report for the OLS estimation of **reduced price equation** :

$$
\begin{aligned}
\widehat{P} = \quad & -32.51 && +1.71PS && +7.60DI && +1.35PF \\
\text{(t)} \quad & (-4.0721) && (4.8682) && (4.4089) && (4.5356) \\
\text{(se)} \quad & (7.9842) && (0.3509) && (1.7243) && (0.2985) \\
\text{(fitness)} n = 30; \quad & R^2 = 0.8887; \bar{R}^2 = 0.8758 \\
& F^* = 69.19; p = 0.0000
\end{aligned}
$$

> **Questions**:
>
> Please explain the regression results??

Example: First OLS estimation on price equation

After OLS estimating of the **reduced price equation**, we can obtain $\hat{P}_i, \hat{v}_{i1}$:

Estimate the price equation and get hat.Pi and hat.vli

P	Q	PS	DI	PF	hat.Pi	hat.vi
29.64	19.89	19.97	2.103	10.52	31.8304	-2.1904
40.23	13.04	18.04	2.043	19.67	40.4658	-0.2358
34.71	19.61	22.36	1.87	13.74	38.5011	-3.7911
41.43	17.13	20.87	1.525	17.95	39.033	2.397
53.37	22.55	19.79	2.709	13.71	40.449	12.921
38.52	6.37	15.98	2.489	24.95	47.4863	-8.9663

Showing 1 to 6 of 30 entries

Previous 1 2 3 4 5 Next

Example: Second OLS estimation on Hausman equation

We proceed to the second step of the endogeneity test:

$$
\begin{aligned}
Q_t &= \beta_0 + \beta_1 \hat{P}_t + \beta_1 \hat{v}_{t1} + u_{t2} && \text{(Hausman equation)} \\
Q_t &= \beta_0 + \beta_1 P_t + \beta_1 \hat{v}_{t1} + u_{t2} && \text{(Pindyck equation)}
\end{aligned}
$$

> Note: you can use one of these two test model:
>
> - Hausman equation for Hausman test
> - Pindyck equation for Pindyck test

Let us do the Hausman test firstly, and then the Pindyck test.

Example: Second OLS estimation on Hausman equation

The raw R report for the **Hausman test equation** show as following:

```
Call:
lm(formula = Hausman_models$mod.Q.Hausman, data = truffles_Hausman)

Residuals:
   Min     1Q Median     3Q    Max 
-7.476 -2.892  0.277  3.394  5.380 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)   11.946      2.604    4.59  9.2e-05 ***
hat.Pi         0.104      0.040    2.60   0.0151 *  
hat.vi         0.338      0.113    2.99   0.0059 ** 
---
Signif. codes:  
0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 3.8 on 27 degrees of freedom
Multiple R-squared:  0.367,	Adjusted R-squared:  0.32 
F-statistic: 7.84 on 2 and 27 DF,  p-value: 0.00207
```

Example: Second OLS estimation on Hausman equation

The tidy R report for the **Hausman test equation** show as following:

$$
\begin{array}{llll}
\widehat{Q} = & +11.95 & +0.10\hat{P} & +0.34\hat{v}_1 \\
(\text{t}) & (4.5880) & (2.5952) & (2.9901) \\
(\text{se}) & (2.6037) & (0.0400) & (0.1130) \\
(\text{fitness}) n = 30; & R^2 = 0.3673; \bar{R}^2 = 0.3205 & & \\
& F^* = 7.84; p = 0.0021 & &
\end{array}
$$

- Conclusions of Hausman simultaneity test:
  - the coefficient of $\hat{v}_{1i}$ is 0.34， $t^* = 2.9901 > t(\alpha/2, n-3) = 2.47$。(given $\alpha = 0.01$)
  - Hence, the t test on coefficient of $\hat{v}_{i1}$ is **significantly**(given $\alpha = 0.01$). Thus we should **reject** H_0, and accept H_1. Finally conclude that there is **no exist** of simultaneity problem。

Example: Second OLS estimation on Pindyck equation

The raw R report for the **Pindyck test equation** show as following:

```
Call:
lm(formula = Hausman_models$mod.Q.Pindyck, data = truffles_Hausman)

Residuals:
   Min     1Q Median     3Q    Max
-7.476 -2.892  0.277  3.394  5.380

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   11.946      2.604    4.59  9.2e-05 ***
P              0.104      0.040    2.60    0.015 *
hat.vi         0.234      0.120    1.95    0.061 .
---
Signif. codes:
0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 3.8 on 27 degrees of freedom
Multiple R-squared:  0.367,    Adjusted R-squared:  0.32
F-statistic: 7.84 on 2 and 27 DF,  p-value: 0.00207
```

Example: Second OLS estimation on Pindyck equation

The tidy R report for the **Pindyck test equation** show as following:

$$
\begin{aligned}
\widehat{Q} = &\quad +11.95 &\quad +0.10P &\quad +0.23\hat{v}_1 \\
\text{(t)} &\quad (4.5880) &\quad (2.5952) &\quad (1.9529) \\
\text{(se)} &\quad (2.6037) &\quad (0.0400) &\quad (0.1199) \\
\text{(fitness)} n = 30; &\quad R^2 = 0.3673; \bar{R}^2 = 0.3205 \\
&\quad F^* = 7.84; p = 0.0021
\end{aligned}
$$

- Conclusions of Pindyck simultaneity test:
  - The coefficient of $\hat{v}_{i1}$ is 0.23, and $t^* = 1.9529 > t(\alpha/2, n-3) = 1.7$. (given $\alpha = 0.1$)
  - Hence, the t test on coefficient of $\hat{v}_{i1}$ is **significantly**(given $\alpha = 0.01$). Thus we should **reject** H_0, and accept H_1. Finally conclude that there is **no exist** of simultaneity problem。

19.4 Exogeneity test*

Because we have learn these techniques in chapter 17, so we will jump to the chapter 20.

Exogeneity test

ellipsis ...

Key points and conclusions

- The problem of identification precedes the problem of estimation.
- The identification problem asks whether one can obtain unique numerical estimates of the structural coefficients from the estimated reduced-form coefficients.
- If this can be done, an equation in a system of simultaneous equations is identified. If this cannot be done, that equation is un- or under-identified.
- An identified equation can be just identified or overidentified. In the former case, unique values of structural coefficients can be obtained; in the latter, there may be more than one value for one or more structural parameters.
- The identification problem arises because the same set of data may be compatible with different sets of structural coefficients, that is, different models. Thus, in the regression of price on quantity only, it is difficult to tell whether one is estimating the supply function or the demand function, because price and quantity enter both equations.

Key points and conclusions

- To assess the identifiability of a structural SEM, one may apply the technique of reduced-form SEM.
- Although the order condition is easy to apply, it provides only a necessary condition for identification. On the other hand, the rank condition is both a necessary and sufficient condition for identification.
- In the presence of **simultaneity**, OLS is generally not applicable But if one wants to use it nonetheless, it is imperative to test for simul-taneity explicitly. The Hausman specification test can be used for this purpose.
- Although in practice deciding whether a variable is endogenous or exogenous is a matter of **judgment**, one can use the Hausman specification test to determine whether a variable or group of variables is endogenous or exogenous.
- Although they are in the same family, the concepts of **causality** and **exogeneity** are different and one may not necessarily imply the other. In practice it is better to keep those concepts separate.

End of this chapter